International Journal of Theoretical Physics, Vol. 34, No. 4, 1995

Dynamical Bifurcation with Noise

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Received June 15, 1994

It was shown by A. Neishtadt that dynamical bifurcation, in which the control parameter is varied with a small but finite speed ϵ, is characterized by a *delay in bifurcation*, here denoted λ_j and depending on ϵ. Here we study dynamical bifurcation, in the framework and with the language of Landau theory of phase transitions, in the presence of a Gaussian noise of strength σ. By numerical experiments at fixed $\epsilon = \epsilon_0$, we study the dependence of λ_j on σ for order parameters of dimension ≤ 3; an exact scaling relation satisfied by the equations permits us to obtain for this the behavior for general ϵ. We find that in the small-noise regime $\lambda_j(\sigma) \simeq a\sigma^{(-b)}$, while in the strong-noise regime $\lambda_j(\sigma) \simeq ce^{(-d)}$; we also measure the parameters in these formulas.

1. INTRODUCTION

Dynamical bifurcation (Neishtadt, 1988a,b), first studied by pure mathematicians in the context of dynamical systems and bifurcation theory (Neishtadt, 1988a,b; Aframovich *et al.*, 1988), has recently received wider attention, not only in the context of mathematics, but also due to its applications, in particular physical ones; see, e.g., the proceedings volume edited by Benoit (1991) and references therein. The reader is referred to Benoit (1991) for a mathematical setting of the theory, and to Lobry (1991) for a general introduction; see also Gaeta (1993) for an application to Landau theory and Landau–Ginzburg-type equations and transitions with nearly degenerate critical modes.

The problem of *delayed bifurcation* in phase transitions with swept order parameter—as dynamical bifurcation is also referred to—is also considered in the physical literature, mainly in connection with laser problems (van der

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0020-7748/95/0400-0595$07.50/0

Broek and Mandel, 1987; Fronzoni *et al.*, 1987; Torrent and San Miguel, 1988; Lythe and Proctor, 1993).

In the present contribution we will discuss dynamical bifurcations with noise (DBN), more specifically in the context of Landau theory of phase transitions (Landau and Lifshitz, 1958) with a scalar or vector order parameter; contrary to previous physical contributions, we will not focus on any concrete realization, but measure some universal characteristics of this phenomenon.

We want to discuss how the picture of dynamical bifurcation (which, in the present context, refers to dynamical phase transitions) is modified in the presence of external noise. In order to do this, we will resort to heuristic considerations, and substantiate them by means of a numerical study of model equations.

We will first briefly recall in Section 2 the main feature of dynamical bifurcation in the language of Landau theory (with scalar order parameter), fix notation, and also recall some standard heuristic reasoning dealing with deterministic dynamical bifurcation.

In Section 3 we will consider the effect of noise, and extend the heuristic reasoning of Section 2 to the stochastic case; this will lead us to expect different regimes, depending on the ratio of the parameters characterizing the stochasticity and the time scale of the deterministic dynamics. The behavior of the stochastic system will also be expected to depend on the dimension of the order parameter.

In Section 4 we will present a numerical simulation of dynamical bifurcation with noise in the case of a scalar order parameter, and confront it with the results expected on the basis of our heuristic reasoning. In Section 5 we consider the case of vector order parameter, and present similar simulations for a vector order parameter of dimension two and three, obtaining a picture similar to the one holding for scalar order parameter.

2. DYNAMICAL BIFURCATIONS

In the Landau theory of phase transitions with a scalar order parameter x, one is faced with a quartic pseudopotential $V(x) = \lambda x^2/2 + x^4/4$, and the dynamics of the system is described by

$$\dot{x} = \lambda x - x^3 \tag{1}$$

so that for $\lambda < 0$ the point $x = 0$ (i.e., the phase it represents) is stable, and for $\lambda > 0$ the stable phase is represented by points $x(\lambda) = \pm\sqrt{\lambda}$. The physical interpretation of this statement is that if we perform an experiment at different values of the control parameter λ and wait long enough for the system to reach equilibrium, then the observed values x_{eq} of the order parameter will be such that $x_{eq}^2(\lambda) = \lambda$.

It was remarked by Neishtadt (1988a,b) that this "long enough" time could—and indeed does—grow beyond any limit in the vicinity of the critical value of λ (this corresponds to critical slowing down of the dynamics). Moreover, in many experimental situations, one does not actually perform separate experiments for different values λ, but instead has a single experiment in which the value of the control parameter λ is varied as slowly as possible, and the value of $x = x(\lambda)$ is observed at different values of λ, i.e., dynamically (this is typically the case in fluid mechanics). For these reasons, it is interesting to consider the case where (1) is substituted by the system

$$\dot{x} = (\lambda - x^2)x \qquad (2)$$
$$\dot{\lambda} = \epsilon$$

Notice that now if x_0 is very small and we start with $\lambda_0 = 0$, for small times the x evolution will be described by $\dot{x} = \epsilon t x$, which gives $x(t) \simeq \exp[\epsilon t^2/2]$, i.e.,

$$x(\lambda) \simeq e^{\lambda^2/(2\epsilon)} \qquad (3)$$

so that we have an explosivelike behavior. Obviously, once x begins to grow, the nonlinear terms counteract the growth, and we only have a jump from a phase $x \simeq 0$ to a phase $x \simeq x_{eq}$. This fact is better illustrated by Fig. 1, where we numerically integrate equation (2).

The value of λ for which this jump happens will be called the "jumping value" λ_j, and the corresponding value of t will be the "jumping time" $t_j = \lambda_j/\epsilon$. These can be defined operationally as the value of λ for which the ratio $\rho(\lambda) = |x(\lambda)/x_{eq}(\lambda)| \leq 1$ is first superior to a given value ρ_0 (for $\lambda > x_0^2$).[3]

This delay in joining the x_{eq} branch is easily understood considering that $dx_{eq}(\lambda)/d\lambda = \dot{\lambda}/\sqrt{\lambda}$, while from (3) we see that for small t, $dx(\lambda)/d\lambda \simeq (\lambda/\epsilon)x_0$, so that $x(\lambda)$ cannot follow $x_{eq}(\lambda)$ at least until the two speeds of variation are not equal, i.e., up to

$$t_j \simeq \left[\frac{\epsilon}{4x_0^2}\right]^{1/3}; \qquad \lambda_j \simeq \left[\frac{\epsilon}{2x_0}\right]^{2/3} \qquad (4)$$

3. DYNAMICAL BIFURCATIONS WITH NOISE

It is quite natural to ask in which way the general description of dynamical bifurcations—and the situation depicted in Fig. 1—is modified in the presence of noise, i.e., if we substitute (2)

$$dx = (\lambda x - x^3)\, dt + \sigma\, dw \qquad (5)$$
$$d\lambda = \epsilon\, dt$$

[3] The restriction $\lambda > x_0^2$ is due to the fact that we need $x_0 \neq 0$ or the system remains trapped in the unstable equilibrium $x = 0$; the condition can be dropped in the noisy case, as then x_0 can be taken to be zero.

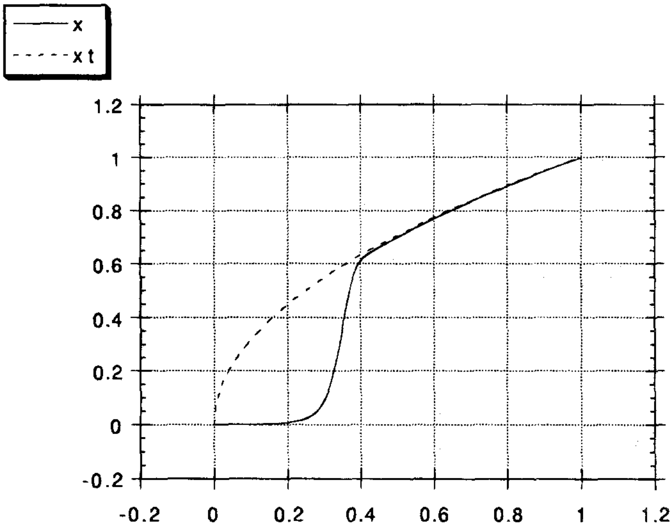

Fig. 1. Dynamical bifurcation versus standard bifurcation. The solid line represents a numerical integration of equation (2), with a parametric plot of x (vertical axis) versus λ, the dashed line the prediction of standard bifurcation theory, i.e., $x = \sqrt{\lambda}$. The jumplike behavior is evident.

where $w(t)$ is the Wiener process with unit variance and the real parameter $\sigma \geq 0$ represents the strength of the noise.

On the basis of the qualitative discussion of the previous section, we expect that, considering $x_0 = 0$, $\lambda_0 = 0$ (notice that $x_0 = 0$ enforces $\lambda_j = \infty$ for $\sigma = 0$) and fixed ϵ, when observing the dependence of λ_j on σ, one would observe a crossover between a small-noise behavior governed by the deterministic dynamics—in which the role of fluctuations is only to drive the system out of $x = 0$—and a strong-noise behavior which is essentially governed by fluctuations, with disappearance of the delay in the bifurcation.

The numerical experiments described below show that both for x scalar and x a vector the small-noise behavior is well described by a function of the form

$$\lambda_j(\sigma) \simeq a\sigma^b \tag{6}$$

while the strong-noise behavior is best described by a function of the form

$$\lambda_j(\sigma) \simeq ce^{-d\sigma} \tag{7}$$

It should be noted that (5)—as well as (2)—has a scaling invariance under

$$x \to \alpha x; \qquad \lambda \to \alpha^2\lambda; \qquad t \to \alpha^{-2}t; \qquad \sigma \to \alpha\sigma \tag{8}$$

so that the behavior observed for a given value of ϵ and varying σ (or vice versa) represents the most general behavior for varying ϵ, σ, through rescaling.

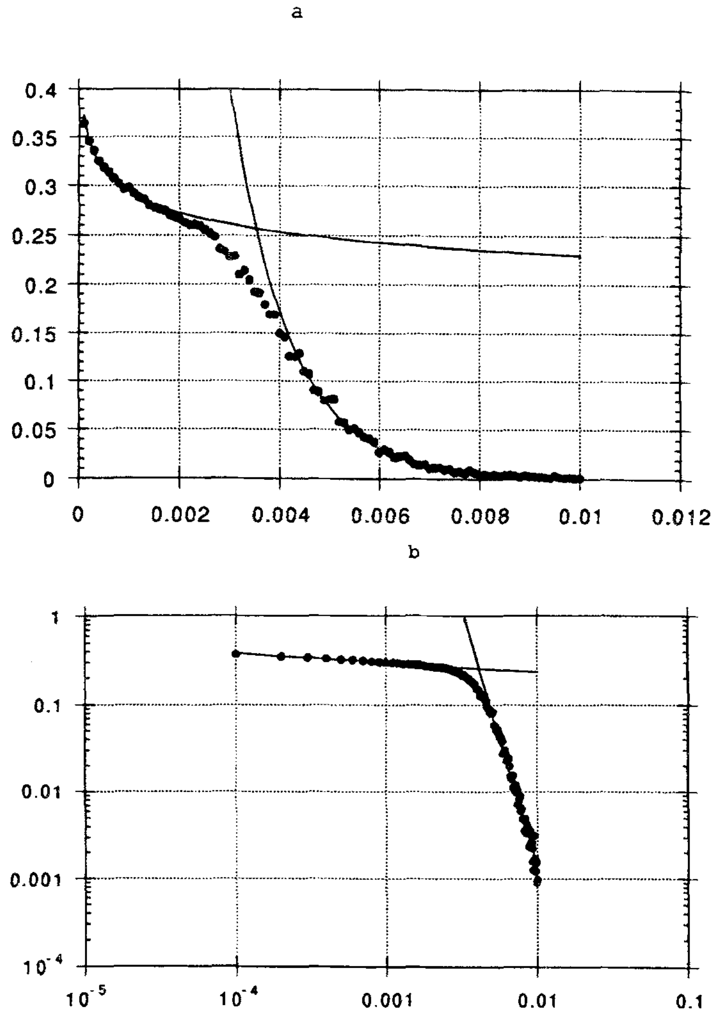

Fig. 2. Dynamical bifurcation with scalar order parameter. For the numerical experiment described in the text, we plot λ_j (vertical axis) as a function of σ. The dots represent the data from numerical simulation, the solid lines the fits for low-noise and high-noise regimes. (a) A linear scale; (b) the data and fits presented in a log-log plot to enhance the different regimes.

We want to check numerically if the above crossover picture, and the claimed behaviors (6) and (7) in the weak- and strong-noise regimes, are indeed correct predictions; we would also like to measure the values of the parameters appearing in (6) and (7).

Although some abstract theorems have been proved for dynamical bifurcation with noise (Benoit, 1991), I have not been able to find any quantitative discussion in the mathematical literature; in the physical literature (see, e.g., van der Broek and Mandel, 1987; Fronzoni *et al.*, 1987; Torrent and San Miguel, 1988; Lythe and Proctor, 1993; and references therein) quantitative analytical discussions are considered, as well as experimental measurements on systems undergoing dynamical bifurcations, but while analytical results in terms of stochastic dynamics (see, e.g., Torrent and San Miguel, 1988) are quite general, it seems that numerical characterization was restricted to specific systems of physical interest. Thus, I believe this to be the first quantitative measurement of the "universal" (model-independent) effects of noise in dynamical bifurcations.

4. NUMERICAL RESULTS

I have integrated numerically equations (5) in several runs, each of them with initial data $x_0 = 0$, $\lambda_0 = 0$ for $\epsilon = 0.01$, and using a time step $\delta t = 0.01$; the terms corresponding to the Wiener process were obtained by a Gaussian random number generator.

The integration was performed over 100 values of σ, uniformly distributed between 0 and 0.01. For each value of σ, I conducted "experiments" made by 100 separate runs of the program, and measured the value of λ for which $\rho(\lambda) \equiv x(\lambda)/x_{eq}(\lambda)$ first became higher than $\rho_0 = 0.8$; this was taken as representing the $\lambda_j(\sigma)$. I conducted five such numerical experiments, which

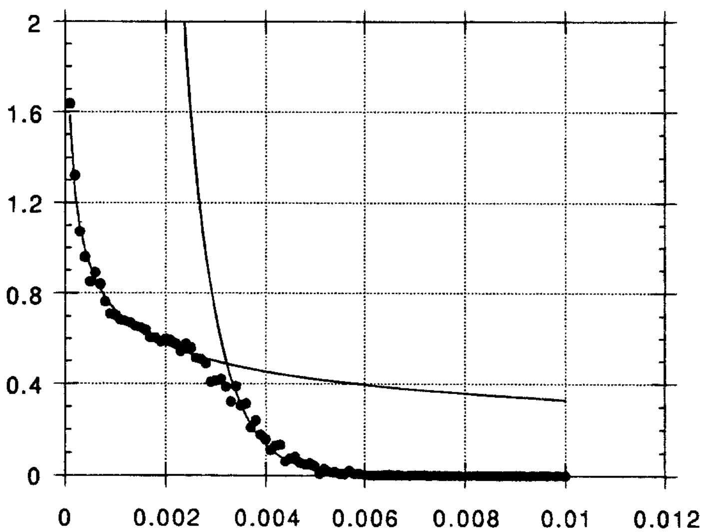

Fig. 3. Dynamical bifurcation with vector order parameter. The data and fits are reported in Fig. 2, for the numerical experiments with two-dimensional order parameter.

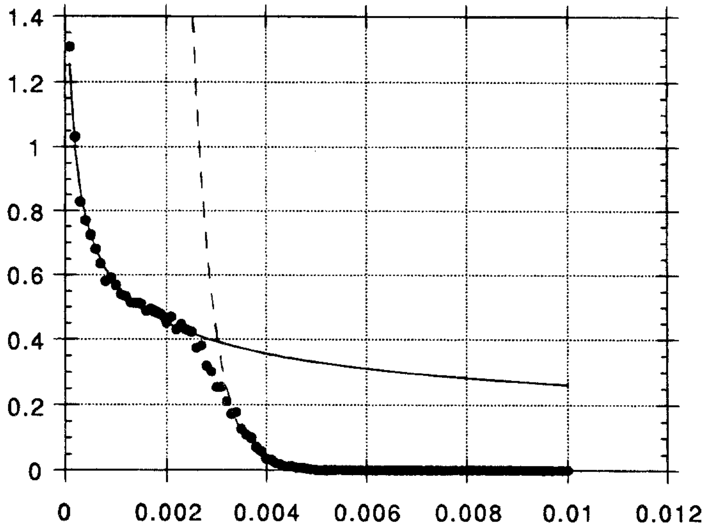

Fig. 4. Dynamical bifurcation with vector order parameter. The data and fits are reported as in Fig. 2, for the numerical experiments with three-dimensional order parameter.

showed very little fluctuations between any two sets of data, suggesting the statistic is satisfactory already within a single experiment, i.e., with 100 runs for each value of σ (see below). The results of this measurement are plotted in Fig. 2, the error bars corresponding to fluctuations of the statistical distribution of λ_j's for each considered σ in different experiments.

I then attempted to fit the data for $\lambda_j(\sigma)$ obtained in this way by functions (6) and (7); i.e., I tried a fit of the form (6) for small σ and (7) for large σ. The functions (6), (7) do indeed provide the best fits to the numerical data among the different functions I tested, and were chosen for this reason.

The best fit, represented by the continuous lines in Fig. 2, was found to correspond to the approximate values

$$a = 0.142, \qquad b = 0.105; \qquad c = 5.178, \qquad d = 851.0 \tag{9}$$

The statistical significance for the weak-noise-regime ($\sigma \leq 0.0020$) fit and the one for the strong-noise-regime ($\sigma > 0.0045$) fit were both approximately 0.99, on the basis of a χ^2 test.

5. VECTOR ORDER PARAMETER

Equation (2) is immediately generalized to the case of a vector order parameter, i.e., if $x \in \mathbf{R}^n$, then x^2 should be understood as (x, x) with $(\cdot, \cdot)$ the standard scalar product in $\mathbf{R}^n$.

Table I. Parameters Providing the Best Fits of Numerical Data as Reported in Figs. 2–4 with Low- and High-σ Regimes Described by Functions of the Form (6) and (7), Respectively

dim	a	b	c	d
1	0.142	0.105	5.178	851
2	0.069	0.340	96.97	1645
3	0.054	0.341	886.8	2552

Similarly, the heuristic considerations presented on (2) also immediately generalize to (2′), so that we expect a similar behavior for the vector case; it should be mentioned that the crossover between the weak-noise and the strong-noise regimes is expected to be located at higher values of σ, for dimensional considerations on Brownian noise trajectories.

I conducted a numerical experiment for the cases $x \in \mathbf{R}^2$ and $x \in \mathbf{R}^3$, with the same setting and procedure as in the scalar case (here one confronts $|x|$ and $\sqrt{\lambda}$); in these cases, only one set of 100 separate runs for each value of σ was conducted, justified by the observations on the statistics in the scalar case. The results of these numerical experiments are given by Figs. 3 and 4, respectively.

The parameters characterizing the fit with functions of the form (6) and (7) are given in Table I, where the corresponding values for the scalar case are also reported again for completeness.

6. CONCLUSIONS

We have shown that the expectation, based on heuristic considerations sketched in our previous discussion, of a crossover between different regimes (weak and strong noise) is well confirmed by the numerical experimental data; moreover, we have been able to fit satisfactorily the numerical data with an inverse power for weak noise and with an exponential for strong noise, as in formulas (6) and (7); and to measure the values of the parameters appearing in (6) and (7). This was done both in the case of scalar order parameter and of vector order parameter of dimension two or three.

As already mentioned, although the numerical experiments were conducted at a fixed value of ϵ, the scaling properties of (5)—given by (8)—extend the validity of these to the case of general ϵ.

We have not conducted any numerical investigation for a vector order parameter of dimension greater than three. We have also not attempted to discuss the form of the functions (6) and (7) describing the different limit regimes, limiting ourselves to a numerical investigation.

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